On the Convergence of an Algorithm Computing Minimum-Norm Solutions of Ill-Posed Problems

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Abstract. The paper studies a finite element algorithm giving approximations to the minimum-norm solution of ill-posed problems of the form $Af = g$, where A is a bounded linear operator from one Hilbert space to another. It is shown that the algorithm is norm convergent in the general case and an error bound is derived for the case where g is in the range of AA^*. As an example, the method has been applied to the problem of evaluating the second derivative f of a function g numerically.

1. **Introduction.** Let X and Y be real Hilbert spaces and A a bounded linear operator from X into Y. The problem of determining, for given g in the range $A(X)$ of A, solutions f in X of

$$(1) \qquad\qquad Af = g$$

is said to be *ill-posed* if A has no continuous inverse on $A(X)$. If there are more than one solutions of (1) one is frequently interested in the solution which has minimum norm in X, the *minimum-norm solution* f_0 of (1). The existence and uniqueness of f_0 is a consequence of the well-known result (see e.g. [6, p. 124]) on the existence of a unique nearest point to the origin of X in every closed convex set in X. Solving (1) for g not in the closure of the range of A, in the sense that the solution f has to satisfy $\|Af - g\| = \min\{ \|Ah - g\| : h \in X\}$, is easily seen to be equivalent to the problem of solving $Af = Pg$, where P is the orthogonal projection from Y onto the closure $\overline{A(X)}$ of $A(X)$ in Y. There are numerous examples of problems of the form (1), e.g. Fredholm integral equations of the first kind, where the mathematical formulation of a host of applications is leading to such equations (see e.g. [13] or [14]).

Usually, ill-posed problems are extremely difficult to solve numerically. Several schemes leading to approximate solutions of (1) have been proposed, e.g. the collocation method (e.g. in [1], [2], [4]) or Tikhonov's method [3], [10], [11], [12]. But up to now, little is known on the convergence of both methods in the general case (e.g. if A does not possess an inverse), as on the crucial choice of regularization parameters occurring in Tikhonov's method. The reason for the lack of a convergence proof for the first method lies in the fact that there are still no results available on the influence of the collocation errors in solving (1) by this method.

In this paper we consider an algorithm for solving (1) numerically. This algorithm, giving the minimum-norm solution of (1), has been proposed in [7], [8]. The algorithm has no restrictions on the operator A (such as the existence of an inverse of A, a condition which is often required for the Tikhonov method; see e.g. [12, p. 45] or [3, p. 870]). Furthermore, the algorithm studied in this article uses a technique whose computational part has some similarities with Galerkin's method, though it is conceptionally much different from Galerkin's and also from Tikhonov's methods. An upper bound for the rate of convergence is derived for the restricted class of problems (1), where A is an arbitrary bounded linear operator and g is in the range of AA^*, A^*: $Y \longrightarrow X$ being the adjoint operator of A. More generally, if g is in the range of A, then a convergence proof for the algorithm follows as a corollary. In case of Hermitian operators A ($X = Y$, $A^* = A$) an upper bound for the error norm in computing the minimum-norm solution of (1) has been derived in [7] for g in the range of A^2 and a convergence proof has been given for the case of arbitrary A and g in [8].

2. Description of the Algorithm. Let $\{P_n\}$ be a sequence of orthogonal projections of finite rank in X such that

$$\lim_n \|P_n f_0 - f_0\| = 0,$$

where f_0 is the minimum-norm solution of (1). The above condition is obviously satisfied if the sequence $\{P_n(X)\}$ is increasing and the union of the sets $P_n(X)$ is dense in X, e.g. if the P_n are the projections associated with an orthonormal basis $\{u_1, \dots, u_n\}$ of X, where $P_n v = \Sigma_{i=1}^n (v, u_i) u_i$, $v \in X$. Several well-known bounds for $\|P_n f_0 - f_0\|$ are available in the literature, e.g.

$$\|P_n f_0 - f_0\| = \mathcal{O}(h^k)$$

if f_0 has an absolutely continuous $(k - 1)$th derivative, the kth derivative of f_0 is in $L_2[a, b]$ and $P_n(L_2[a, b])$ is the set of polynomial spline functions of odd degree $\geqslant k - 1$ with knots $a, a + h, a + 2h, \dots, b - h, b$ (see e.g. [9], where a proof is given for the case $k = 2$ and 4).

If the sequence $\{a_n\}$ is given by

(2a) $a_n = \inf\{\|Af - g\|: f \in P_n(X)\}$

and the sequence of positive numbers $\{b_n\}$ is chosen such that

(2b) $\|P_n f_0 - f_0\| = o(b_n)$ and $b_n = o(1),$

then an algorithm for solving (1) is established by the construction of a sequence $\{f_n\}$ such that

(2c) $f_n \in S_n$ and $\|f_n\| = \inf\{\|f\|: f \in S_n\},$

where S_n is the intersection of the sets $P_n(X)$ and $f_0 + (a_n + b_n)A^{-1}(U)$ and U is the unit ball of X. The existence of a unique element f_n again is a consequence of the fact that each S_n is a closed convex set in X, having a unique nearest point to the

origin of X. If $\{v_1, \ldots, v_{m(n)}\}$ is any basis for $P_n(X)$ it is clear that the vector x_n of basis coefficients $x_{n1}, \ldots, x_{nm(n)}$ of f_n can be computed by evaluating

$$a_n = \inf\{ [y^T B_n y - 2w_n^T y + (g, g)]^{\frac{1}{2}}, y \in \mathbf{R}^{m(n)} \}$$

and by solving

$$x_n^T M_n x_n = \inf\{x_n^T M_n x_n : x_n \in \mathbf{R}^{m(n)}, x_n^T B_n x_n - 2w_n^T x_n + (g, g) = (a_n + b_n)^2 \}$$

for x_n, where M_n is the Gramian $[(v_i, v_j)]$ of the above basis and B_n and w_n are given by

$$B_n = [(Av_i, Av_j)], \qquad w_n = ((Av_1, g), \ldots, (Av_{m(n)}, g))^T.$$

We recognize B_n as the so-called stiffness matrix of the finite element method. It is easy to verify that the matrix $B_n + \lambda M_n$ is (Hermitian and) positive definite for all positive numbers λ. Now, it follows that x_n is the solution x of the system of linear equations

$$(3a) \qquad\qquad\qquad (B_n + \lambda M_n)x = w_n,$$

obtainable by the Cholesky method, where λ is the unique positive solution [7] of the scalar equation

$$(3b) \qquad\qquad -\lambda x^T M_n x - w_n^T x + (g, g) - (a_n + b_n)^2 = 0.$$

Here, Newton's method is applicable, since the derivative of the left-hand side of the above equation, as a function of λ, is $2\lambda x^T M_n (B_n + \lambda M_n)^{-1} M_n x$ and this expression, being the 2λ multiple of the square of the Euclidean norm of an $m(n)$-vector, is non-negative.

Summing up, the sequence $\{f_n\}$ may be computed numerically by solving (3) for x and λ and then taking $f_n = \Sigma_{i=1}^{m(n)} x_{ni} v_i$, where $x_n = x$.

3. An Upper Bound for the Convergence Rate. For a large class of problems the following theorem shows that the ultimate convergence rate for solving (1) by the algorithm (2) is almost of the order of the square root of the error norm of the best approximation in $P_n(X)$ to the minimum-norm solution f_0 of (1).

THEOREM. *For every g in $AA^*(Y)$ the error norm for the algorithm (2) has the property that*

$$\|f_n - f_0\| = O(b_n^{\frac{1}{2}}),$$

where f_0 is the minimum-norm solution of (1).

Proof. Since by (2b) $\|P_n f_0 - f_0\| = o(b_n)$ there is, in view of (1), an integer m such that

$$\|AP_n f_0 - g\| \leqslant \|A\| \|P_n f_0 - f_0\| \leqslant a_n + b_n, \qquad n \geqslant m.$$

Thus

$$P_n f_0 \in f_0 + (a_n + b_n)A^{-1}(U), \qquad n \geqslant m,$$

which by (2c) implies that

(4) $\|f_n\| \leqslant \|P_n f_0\| \leqslant \|f_0\|, \quad n \geqslant m.$

Since $Af_0 = g = AA^*h$ for some h in Y and since by a known theorem [5] f_0 is in the closure $\overline{A^*(Y)}$ in X of the range of A^*, one has

$$f_0 - A^*h \in \overline{A^*(Y)} \cap A^{-1}(0).$$

It is well known that the above intersecting sets are orthogonal. Therefore,

$$f_0 = A^*h.$$

By a twofold application of the Pythagorean theorem it is now easy to see that

(5) $\|f_n - f_0\|^2 \leqslant 2(f_0 - f_n, f_0).$

By (2c) and (2a) this yields

$$
\begin{aligned}
\tfrac{1}{2}\|f_n - f_0\|^2 &\leqslant \sup\{(f, f_0): Af \in (a_n + b_n)U\} \\
&= (a_n + b_n)\sup\{(Af, h): Af \in U\} \\
&\leqslant \|h\|[b_n + \inf\{\|A(f - f_0)\|: f \in P_n(X)\}] \\
&\leqslant \|h\|(b_n + \|A\|\|P_n f_0 - f_0\|), \quad n \geqslant m.
\end{aligned}
$$

Due to (2b) one then has the estimate

$$\|f_n - f_0\|^2 \leqslant 2\|h\|b_n(1 + o(1)), \quad n \geqslant m,$$

hence

$$\|f_n - f_0\| = O(b_n^{1/2}). \quad \square$$

COROLLARY. *For every g in $A(X)$ the sequence $\{f_n\}$ generated by the algorithm (2) converges in X to the minimum-norm solution f_0 of (1).*

Proof. Let $\{f_{k(n)}\}$ be any subsequence of $\{f_n\}$ given by (2). Since by (4), $f_n \in U$, $n \geqslant m$ and since U is weakly sequentially compact in X, there is a subsequence of $\{f_{k(n)}\}$, which we also denote by $\{f_{k(n)}\}$, converging weakly to an element, say f of U. Clearly (5) shows that

(6) $\displaystyle\limsup_n \|f_{k(n)} - f_0\| \leqslant 2(f_0 - f, f_0).$

Next, using (2c), (2a) and (2b), one obtains

$$
\begin{aligned}
(f_0 - f, A^*h) &= \lim_n (f_0 - f_{k(n)}, A^*h) \\
&= \lim_n (g - Af_{k(n)}, h) \leqslant \|h\|\lim_n (a_n + b_n) \\
&\leqslant \|h\|\lim_n (\|A\|\|P_n f_0 - f_0\| + b_n) = 0, \quad h \in Y.
\end{aligned}
$$

Since f_0 is in the closure of $A^*(Y)$ in X it then follows that

$$(f_0 - f, f_0) = 0$$

and finally by (6) that

$$\lim_n \|f_{k(n)} - f_0\| = 0.$$

Finally, since we have shown that any subsequence of $\{f_n\}$ has a subsequence which is norm convergent to f_0, the proof of the corollary is complete. $\square$

4. The Example of Computing the Second Derivative.

The second derivative f of a function g satisfies the well-known Fredholm integral equation of the first kind

$$\int_0^s (s - t) f(t) \, dt = g(s) - g(0), \qquad s \geqslant 0.$$

Defining the integral operator K from $L_2[0, 1]$ into itself with kernel

$$k(s, t) = \begin{cases} s - t, & t \leqslant s, \\ 0, & t > s, \end{cases}$$

the above integral equation obviously has the form (1), $Kf = g$, for all g with $g(0) = 0$. It is clear that K is compact. The following argument shows that K is injective on $L_2[0, 1]$:

Let f be in $K^{-1}(0)$. Then, since f is integrable on $[0, 1]$, $(Kf)(s) = 0$, $s \in [0, 1]$, and for s in $(0, 1)$ and sufficiently small h,

$$\int_0^s f(t) \, dt = h^{-1} \left[(Kf)(s + h) - \int_s^{s+h} (s + h - t) f(t) \, dt - (Kf)(s) \right]$$

$$= -\int_s^{s+h} h^{-1}(s + h - t) f(t) \, dt.$$

This implies

$$\left| \int_0^s f(t) \, dt \right| \leqslant \lim_{h \to 0} \int_s^{s+h} |f(t)| \, dt = 0, \qquad s \in (0, 1),$$

and thus $f = 0$ a.e. on $[0, 1]$.

Now, since K is injective, there is only one solution of $Kf = g$ for every g in the range of K. Therefore, choosing

$$g(s) = (s^4 - 4s^3 + 6s^2)/24, \qquad 0 \leqslant s \leqslant 1,$$

the function f given by

$$f(t) = (t^2 - 2t + 1)/2, \qquad 0 \leqslant t \leqslant 1,$$

is the minimum-norm solution f_0 of $Kf = g$.

For the numerical computations we assume $P_n(L_2[0, 1])$ to be the set of spline functions of degree one with knots $0, 1/n, 2/n, \ldots, 1$ with a basis $\{v_1, \ldots, v_{n+1}\}$ of B-splines satisfying $v_{i+1}(j/n) = \delta_{ij}$ (the Kronecker symbol), $0 \leqslant i$, $j \leqslant n$. The advantage of such a basis (of finite elements) is the minimality of the

support of each v_i (being of length $\leqslant 2/n$) which facilitates the numerical integration of the elements of B_n and w_n. Here, the mass matrix M_n is tridiagonal and can be evaluated exactly, the ijth elements of $6nM_n$ are 1 for $i \neq j$, 2 for $i = j = 1$ or $n + 1$ and 4 for $2 \leqslant i = j \leqslant n$. For the evaluation of Kv_i at desired points, use has been made of two-point Gaussian quadrature on each subinterval of length $1/n$. On the other hand, the scalar products occurring in the definition of B_n and w_n have been computed by the trapezoidal rule with nodes coinciding with the above $n + 1$ knots. The basis $\{v_1, \ldots, v_{n+1}\}$ has the property that $\|P_n f_0 - f_0\| = O(n^{-2})$. In general, if g is in the range of KK^* (the hypothesis of the convergence theorem in Section 3) or in the range of K^2, then we still have the same property. Therefore, it is justified to take, e.g. $b_n = 10^{-3} n^{-2} \log n$, where n runs over the higher powers of 2. The expected convergence rate based on this choice of b_n is, according to the theorem in Section 3, $O(n^{-1} \log n)$. Finally, the following Table 1 gives the error norms $\|f_n - f_0\|$ for a number of approximations f_n for f_0, computed by algorithm (2), i.e. by its numerical variant described in Section 2. Table 1 also indicates the ratio of $\|f_n - f_0\|$ to $n^{-1} \log n$.

<div align="center">TABLE 1</div>

n	$\|f_n - f_0\|$	$\|f_n - f_0\|/(n^{-1} \log n)$
4	$2.1E - 3$	$6.0E - 3$
8	$1.0E - 3$	$3.8E - 3$
16	$4.9E - 4$	$2.8E - 3$
32	$2.3E - 4$	$2.1E - 3$
64	$1.1E - 4$	$1.7E - 3$

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